Fuzzy Space-Time

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The mathematical theory of fuzzy sets is applied to construct a fuzzy Minkowski space-time. An event in this space-time is defined to be a function $\chi: M \to [0, 1]$, where M is the ordinary Minkowski space-time. The notion of fuzzy causal structure is defined.

1. In relativistic physics events are always sharply defined and (in Special Relativity) identified with points of the 4-dimensional Minkowski manifold. However, an event in space-time operationally can be defined as a collision between particles, and the quantum character of the collision makes it impossible to localize the collision sharply. As argued by Wigner [1], the smallest space-time volume $V_{\min}$ to which the collision of two quantum particles can be confined is (apart from a numerical constant of the order of unity)

$$V_{\min} = \frac{\hbar^4 c^3}{E^{5/2}(E + m c^2)^{3/2}},$$

where E is the average kinetic energy of the particles in the reference system in which their centre of mass is at rest. The same formula is valid in the case of two particles of equal mass m, and in the case that either one or both particles have zero rest-mass. This shows that, from the *physical* point of view, events are essentially fuzzy entities. It turns out that the mathematical theory of fuzzy sets [2, 3, 4] is almost ready for physical applications.

In this note we shall develop the idea of the fuzzy Minkowski space-time. At the present stage of its construction the presented model is somewhat naive, in the sense that we are discussing rather a theoretical status of new concepts than their direct physical interpretation.

2. Let X be our universal set or space. A *fuzzy subset* of X is a function

$$\chi: X \to [0, 1].$$

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The real number $\chi(x) \in [0, 1]$, $x \in X$, may be interpreted as being a measure of the probability that the element x is a member of a given subset of X. If $\chi: X \to \{0, 1\}$, we recover the ordinary concept of a subset of X. Operations on fuzzy sets, fuzzy relations, fuzzy functions, etc. can be consistently defined [4].

Let $\mathscr{F}(X)$ be the set of all fuzzy sets on X. We define the function

$$H: \mathscr{F}(X) \to [0, \infty]$$

by

$$H(\chi) = -k \int_X \chi(x) \ln \chi(x) \, \mathrm{d}x,$$

where $x \in X$, and k is a positive constant. $H(\chi)$ is called the *entropy* of the fuzzy set χ, and it may be thought of as a measure of the degree of fuzziness of χ. In the limiting case, for a sharp (ordinary) subset of X, we have

$$H(\chi) = 0, \quad \text{and} \quad \chi(X) = \{0, 1\}.$$

3. Let $X = M = R^4$ be the ordinary Minkowski space-time. We shall call it the Minkowski reference space. We define a *fuzzy event* A to be a function

$$\chi_{\mathrm{A}}: M \to [0, 1],$$

and we say that the probability that the event A occurs at a point $x \in M$ is $\chi_{\mathrm{A}}(x)$. The *fuzzy Minkowski space-time* is assumed to be the set $\tilde{M} \subset \mathscr{F}(M)$ of functions $\chi(x)$ with compact support in M. In a realistic physical situation this assumption might turn out to be too strong. We assume also that $\chi(x)$ is normalized in the following way:

$$\int_M \chi(x) \, \mathrm{d}^4 x = 1.$$

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4. Let $L: x \to x'$, $x, x' \in M$, be an ordinary Poincaré (Lorentz) transformation on the reference space M. We define a *fuzzy Poincaré* (*Lorentz*) *transformation* on $\mathscr{F}(M)$:

$$\mathscr{L}: \chi_A \rightsquigarrow \chi_B, \quad \chi_A, \chi_B \in \mathscr{F}(M)$$

by

$$\mathscr{L}(\chi_A(x)) = \chi_A(L(x)).$$

It is obvious that the transformations $\mathscr{L}$ form a group on $\mathscr{F}(M)$.

5. Let $\chi \in \mathscr{F}(X)$, where X is a universal set. Sets of the form

$$N_\alpha(\chi) = \{x \in X: \chi(x) \geqq \alpha\}, \quad \alpha \in [0, 1],$$

are called *α-cuts* of χ. Let us remark that

$$\alpha_1 < \alpha_2 < \cdots \Rightarrow N_{\alpha_1}(\chi) \supset N_{\alpha_2}(\chi) \supset \cdots,$$

and that the set of all α-cuts of χ: $\{N_\alpha(\chi)\}_{\alpha \in [0,1]}$ is a decreasing sequence.

In terms of α-cuts a *fuzzy causal structure* can be defined on $\tilde{M}$ in the following way:

A fuzzy event χ_A *γ-chronologically precedes* a fuzzy event χ_B, $\chi_A \underset{\gamma}{\ll} \chi_B$, if

$$N_\alpha(\chi_B) \subseteqq I^+(N_\alpha(\chi_A))$$

for every $\alpha \geqq \gamma$; $\alpha, \gamma \in [0, 1]$, where $I^+(N_\alpha(\chi_A))$ is the chronological future of $N_\alpha(\chi_A)$ in the ordinary sense.

A fuzzy event χ_A *γ-causally precedes* a fuzzy event χ_B, $\chi_A \underset{\gamma}{<} \chi_B$, if

$$N_\alpha(\chi_B) \subseteqq J^+(N_\alpha(\chi_A))$$

for every $\alpha \geqq \gamma$; $\alpha, \gamma \in [0, 1]$, where $J^+(N_\alpha(\chi_A))$ is the causal future of $N_\alpha(\chi_A)$ in the ordinary sense.

Time reversal of these relations is defined in a natural way. If two fuzzy events are neither γ-chronologically nor γ-causally connected they are said to be acausal with respect to each other.

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